A fundamental limit in the capability of Gaussian systems in quantum metrology

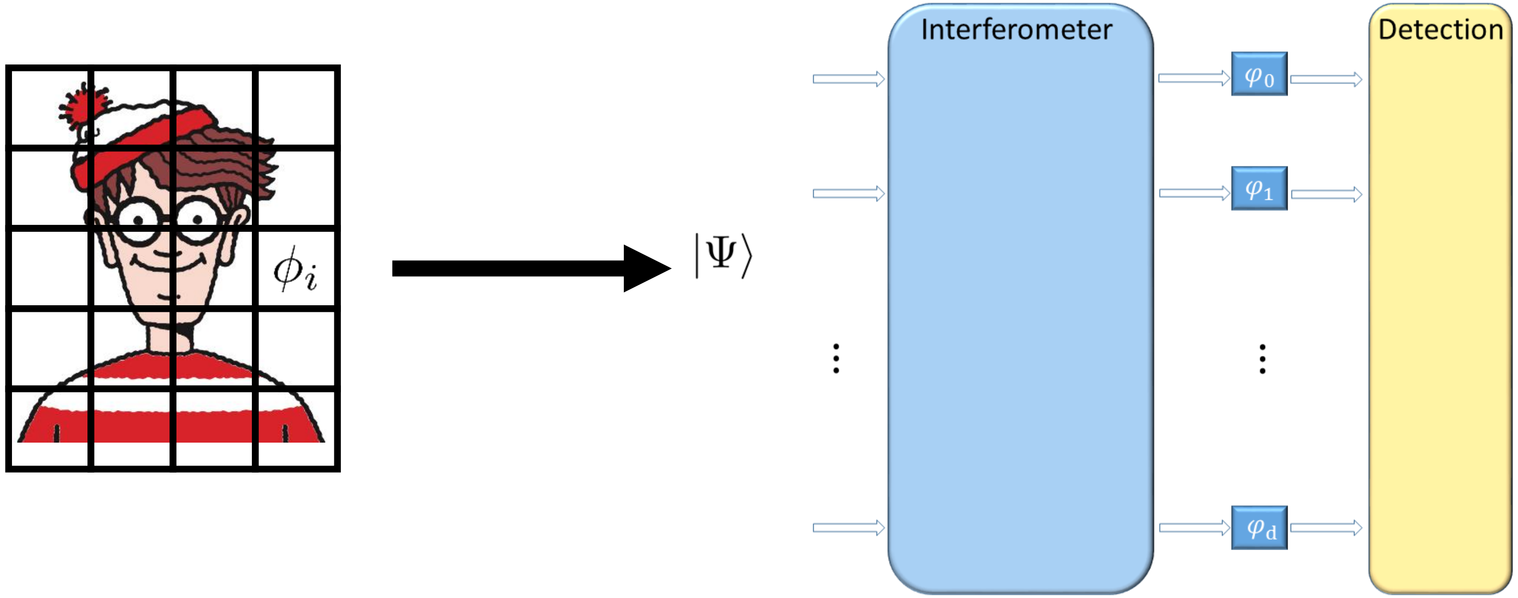
Christos Gagatsos, Dominic Branford, Animesh Datta

University of Warwick, UK

Malta/QTSpace/29.03.2017

Introduction and motivation

The problem: we want to read the pixels of an image. The unknown phases label each pixel.



The performance of any estimation process is captured by the covariance matrix of the estimator $V(\boldsymbol{\varphi})$.

Lower bound: $V(\boldsymbol{\varphi}) \geq \mathrm{H}^{-1}$

Where H is the Quantum Fisher Information Matrix (QFIM)

Bound of the total variance of all the parameters:

$$Tr(V(\boldsymbol{\varphi})) \geq Tr(\mathrm{H}^{-1})$$

- The unknown phases are relative to some reference phase: we cannot measure global phases.
- How the reference mode should be defined?
- Is it better to estimate the phases simultaneously or individually (**restriction:** with same mean energy)?

Bures distance: $d_B(\hat{\rho}(\mu), \hat{\rho}(\mu + d\mu)) = 2\left[1 - \mathcal{F}(\hat{\rho}(\mu), \hat{\rho}(\mu + d\mu))\right]$

Uhlmann fidelity: $\mathcal{F}(\hat{\rho}(\mu), \hat{\rho}(\mu + d\mu)) = \mathrm{Tr}\left[\left(\hat{\rho}(\mu)^{1/2}\hat{\rho}(\mu + d\mu)\hat{\rho}(\mu)^{1/2}\right)^{1/2}\right]$

Second-order terms of Bures wrt to dμ, i.e. the Hessian matrix, is the Quantum Fisher information matrix (with factor of 4) $\mathbf{H}$

$\hat{\mathcal{L}}_i$: SLDs

$$\frac{\partial \hat{\rho}_\phi}{\partial \phi_i} = \frac{\hat{\mathcal{L}}_i \hat{\rho}_\phi + \hat{\rho}_\phi \hat{\mathcal{L}}_i}{2}$$

$$\mathbf{H}_{i,j} = \frac{1}{2}\mathrm{Tr}\left[\hat{\rho}_\phi\left(\hat{\mathcal{L}}_i\hat{\mathcal{L}}_j + \hat{\mathcal{L}}_j\hat{\mathcal{L}}_i\right)\right]$$

Formulae for Gaussian probe states have been found:

PRL 115, 260501 (2015) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2015

Quantum Fidelity for Arbitrary Gaussian States

Leonardo Banchi,[1] Samuel L. Braunstein,[2] and Stefano Pirandola[2]
[1]*Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom*
[2]*Computer Science and York Centre for Quantum Technologies, University of York, York YO10 5GH, United Kingdom*
(Received 9 July 2015; published 22 December 2015)

Formulae for pure probe states $|\Psi\rangle$ are known:

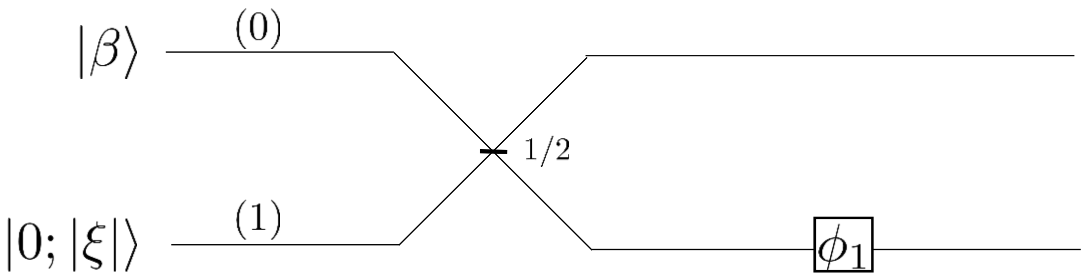
$$H_{ij} = 4Re\left(\langle\partial_i\Psi|\partial_j\Psi\rangle - \langle\partial_i\Psi|\Psi\rangle\langle\Psi\partial_j\Psi\rangle\right)$$

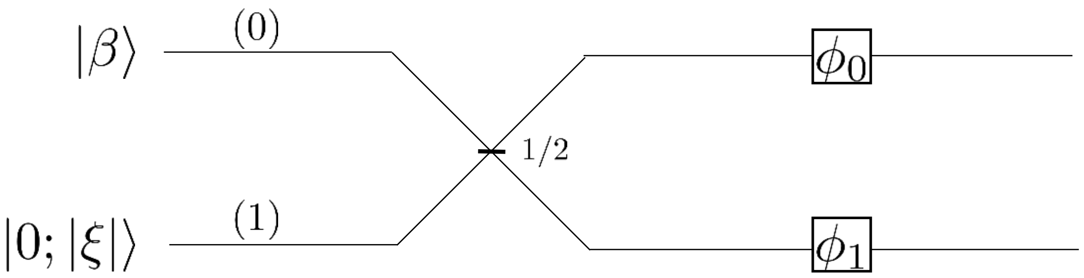
If the generators $\hat{g}_i$ of the parameters commute:

$$H_{ij} = 4\left(\langle\hat{g}_i\hat{g}_j\rangle - \langle\hat{g}_i\rangle\langle\hat{g}_j\rangle\right)$$

The technical difficulties of multiple parameter estimation are:

- Calculating the QFIM for arbitrary d.
- Inverting the QFIM so that it used in $V(\boldsymbol{\varphi}) \geq \mathrm{H}^{-1}$





$$\hat{\mathcal{U}}_\phi = e^{i\phi_1 \hat{n}_1}$$

$$H = (1 + e^{2|\xi|})|\beta|^2 + (2 + \cosh 2|\xi|)\sinh^2|\xi|$$

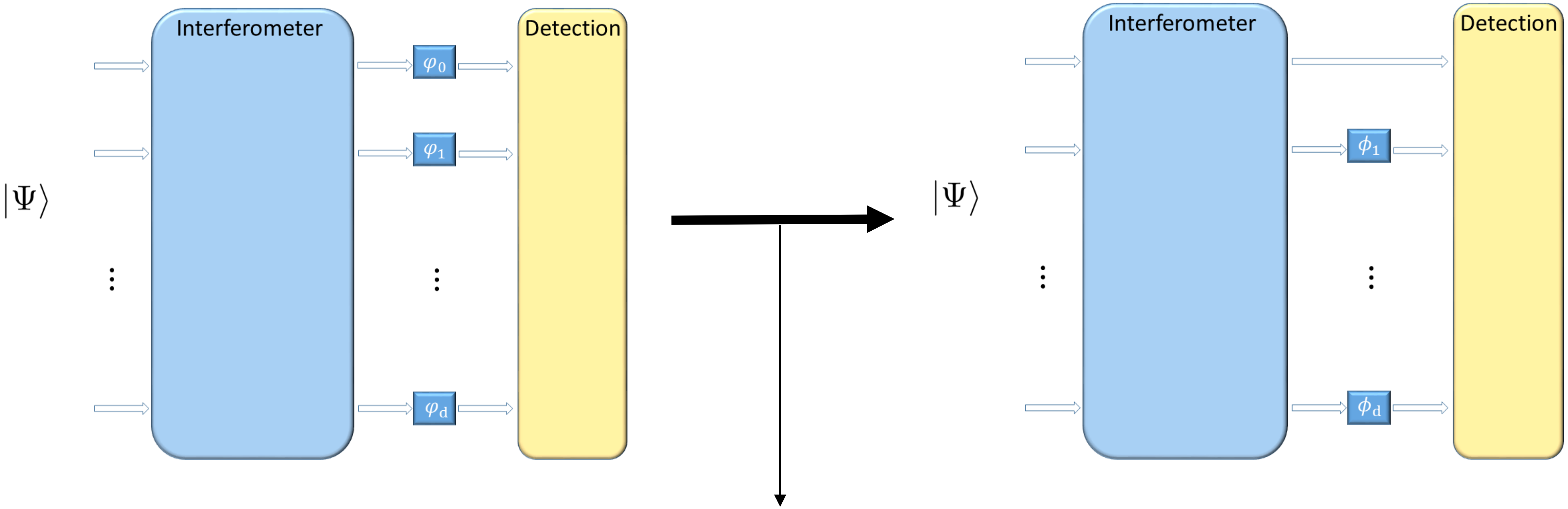
wrong

$$\hat{\mathcal{U}}_\phi = e^{-i\underbrace{(\phi_0+\phi_1)}_{\text{Unmeasurable phase}}\hat{n}_0} e^{i\phi_0\hat{n}_0 + i\phi_1\hat{n}_1} = e^{i\phi_1(\hat{n}_1 - \hat{n}_0)}$$

$$H = 4(|\beta|^2 e^{2\xi} + \sinh^2|\xi|)$$

correct

Reference mode and SU(n) algebra



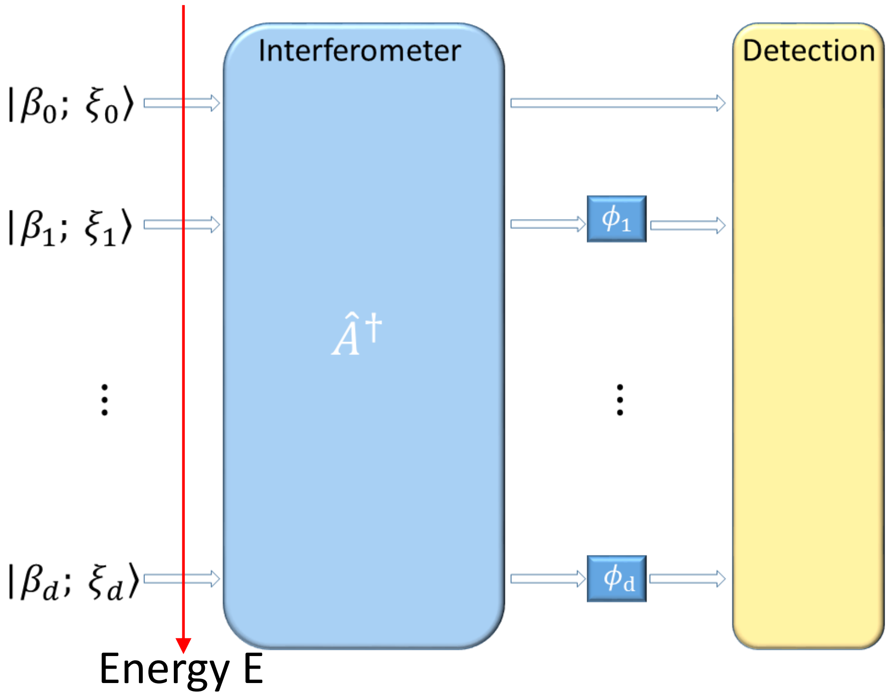
$$\hat{\mathcal{U}}_{\boldsymbol{\phi}} = \hat{\mathcal{U}}'_{\boldsymbol{\phi}} \exp(-i\varphi\hat{n}_0) = \exp(i\phi_1(\hat{n}_1 - \hat{n}_0) + \ldots + i\phi_d(\hat{n}_d - \hat{n}_0)) = \exp\left(i \sum_{i=1}^{d} \phi_i \underbrace{(\hat{n}_i - \hat{n}_0)}_{\hat{g}_i}\right)$$

By setting the reference mode's phase may lead to wrong results. The proper way is to keep the whole problem in SU(n)

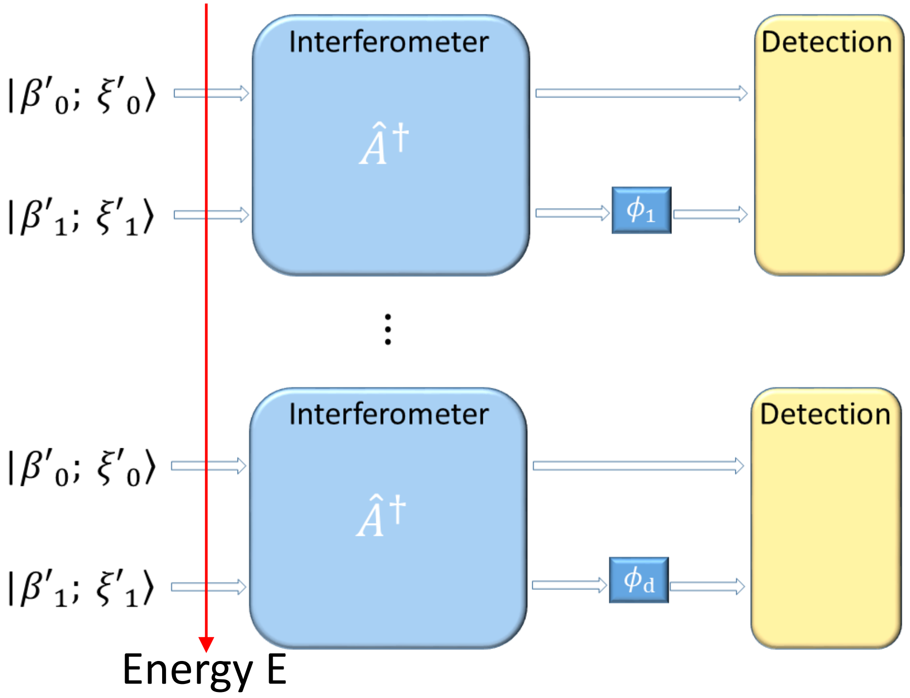
$$\varphi = \phi_0 + \ldots + \phi_d$$

Captures an unmeasurable overall phase

General pure Gaussian input and the comparison we intend to do



$$R = \frac{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{sim}}^{-1}\right)}{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{ind}}^{-1}\right)}$$



$$H_{i,j} = 4(\langle \hat{g}_i \hat{g}_j\rangle - \langle \hat{g}_i\rangle\langle \hat{g}_j\rangle) = 4(h_{i,j} - h_{i,0} - h_{0,j} + h_{0,0})$$

$$G(\boldsymbol{\mu}) = \exp\left[\frac{\boldsymbol{r}_b^\dagger \mathbf{M}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^\dagger \mathbf{M}^{-1}\boldsymbol{r}_b + \boldsymbol{\mu}^\dagger \mathbf{M}^{-1}\boldsymbol{\mu}}{4}\right] \qquad \mathbf{M}^{-1} = 2\begin{pmatrix} \mathbf{E} & -\mathbf{N}\mathbf{E}^T \\ -\mathbf{N}^\dagger\mathbf{E} & \mathbf{E}^T \end{pmatrix}$$

$$\boldsymbol{r}_b = (b_0, \dots, b_d, b_0^*, \dots, b_d^*)^T \equiv (\boldsymbol{b}, \boldsymbol{b}^*)^T$$

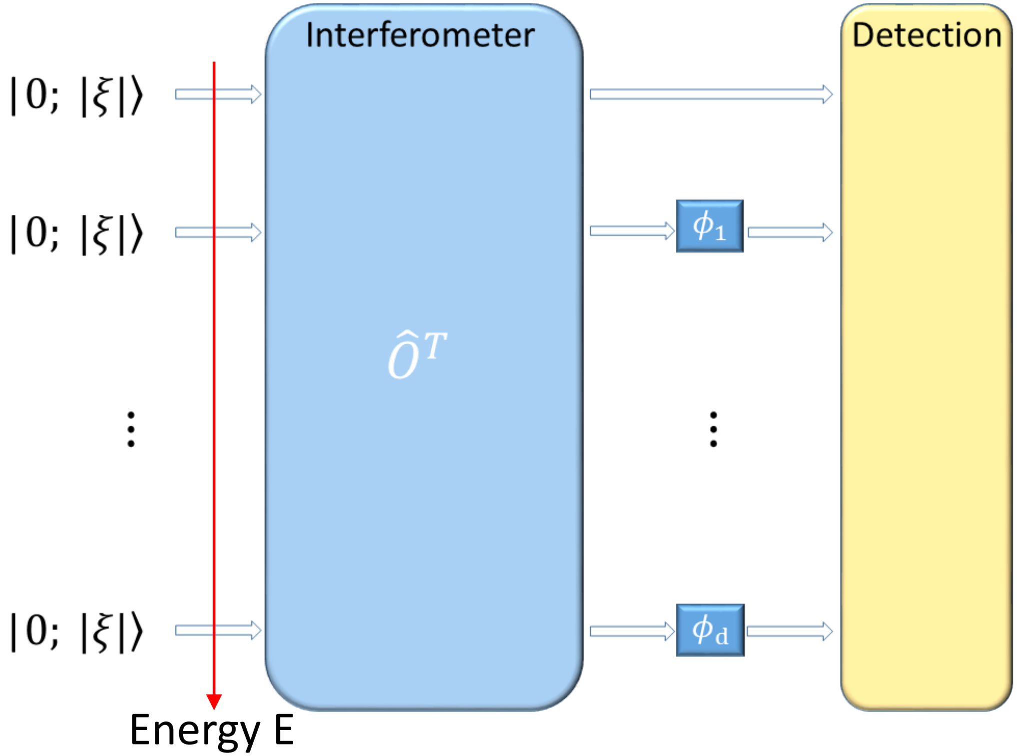
$$b_j = \sum_{k=0}^{d} A_{kj}^* \left(\beta_k + \beta_k^* \tanh|\xi_k|\right)$$

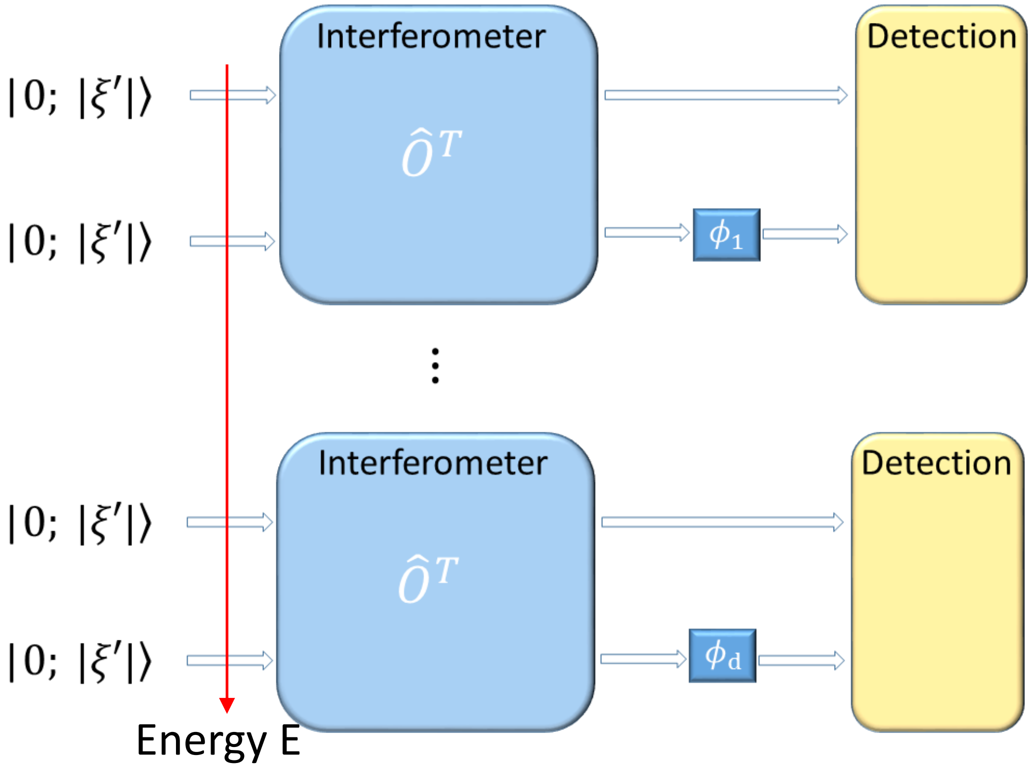
$$\boldsymbol{\mu} = (\lambda_0, \dots, \lambda_d, \lambda_0^*, \dots, \lambda_d^*)^T$$

Derivatives wr.t. to mu

$$h_{i,j} = 4\Big((\mathbf{E}\mathbf{N} - \boldsymbol{\gamma}\boldsymbol{\gamma}^T)\circ(\mathbf{E}\mathbf{N} - \boldsymbol{\gamma}\boldsymbol{\gamma}^T)^* - (\boldsymbol{\gamma}\boldsymbol{\gamma}^T)\circ(\boldsymbol{\gamma}\boldsymbol{\gamma}^T)^* + \frac{1}{4}(\mathbf{E}+\mathbf{E}^*)\circ(\mathbf{E} + \mathbf{E}^* + 2\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger + 2\boldsymbol{\gamma}^*\boldsymbol{\gamma}^T) - (\mathbf{E} + \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)\circ\mathbf{I}\Big)_{i,j},$$

Assumptions and optimization

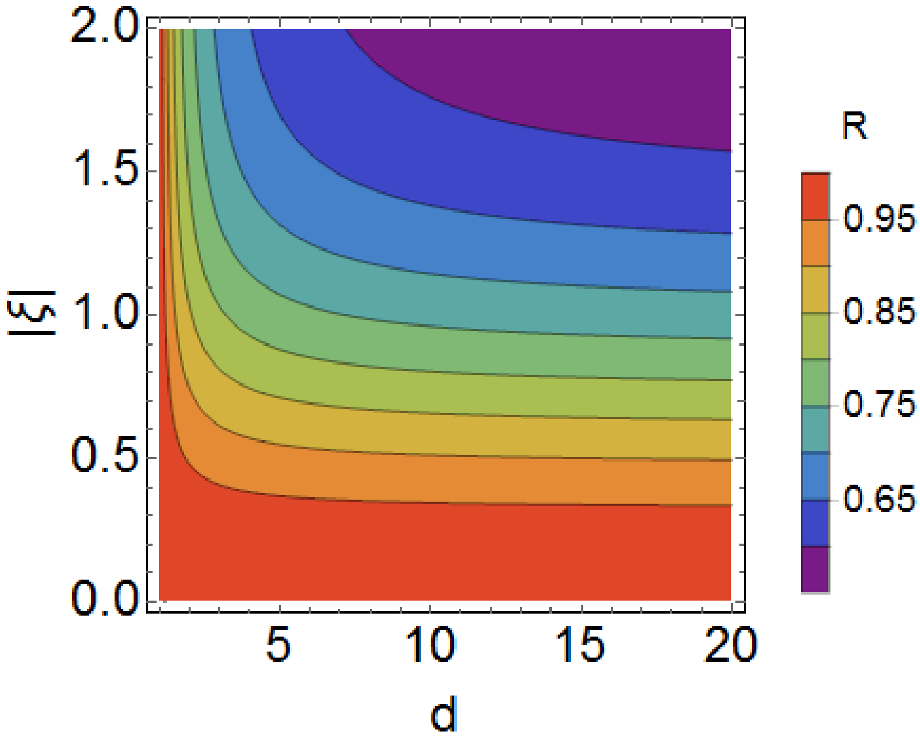




Numerics: Orthogonal transformation and same squeezing

Optimization (Lagrange multipliers)**:** All energy into squeezing

Agrees with recent results: arxiv.org/abs/1601.05912 P. A. Knott et al.



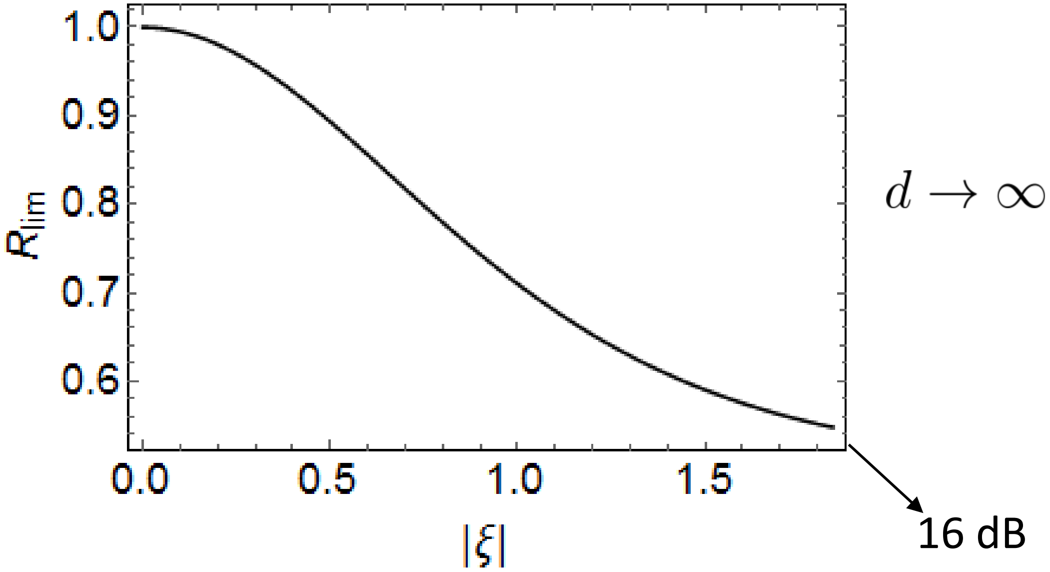
$$R = \frac{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{sim}}^{-1}\right)}{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{ind}}^{-1}\right)} = 1 - \frac{d-1}{2d}\tanh^2|\xi| \leq 1$$

Enhancement because:

- utilizing squeezed states
- one reference mode in the simultaneous strategy

$$R \geq \tfrac{1}{2} \qquad \lim_{d\to\infty}\lim_{|\xi|\to\infty} R = \frac{1}{2}$$

The ratio doesn't go to zero because Gaussian states are not the best choice for estimation (even though very practical).



$$d \to \infty$$

$$R_{\mathrm{lim}} = \lim_{d\to\infty} R = 1 - \frac{1}{2}\tanh^2|\xi|$$

For (almost) realistic squeezing the lower bound can be achieved

Comparison of simultaneous estimation: Gaussian v non-Gaussian states

Gaussian probe states:

$$R \geq \tfrac{1}{2} \qquad \lim_{d\to\infty} \lim_{|\xi|\to\infty} R = \frac{1}{2}$$

GNS (**non-Gaussian**) probe states: $|\Psi\rangle = \frac{1}{1+\sqrt{d}}|N,0,\ldots,0\rangle + \frac{1}{\sqrt{d+\sqrt{d}}}|0,N,0,\ldots,0\rangle + |0,0,\ldots,N\rangle$

d: number of phases to be estimated

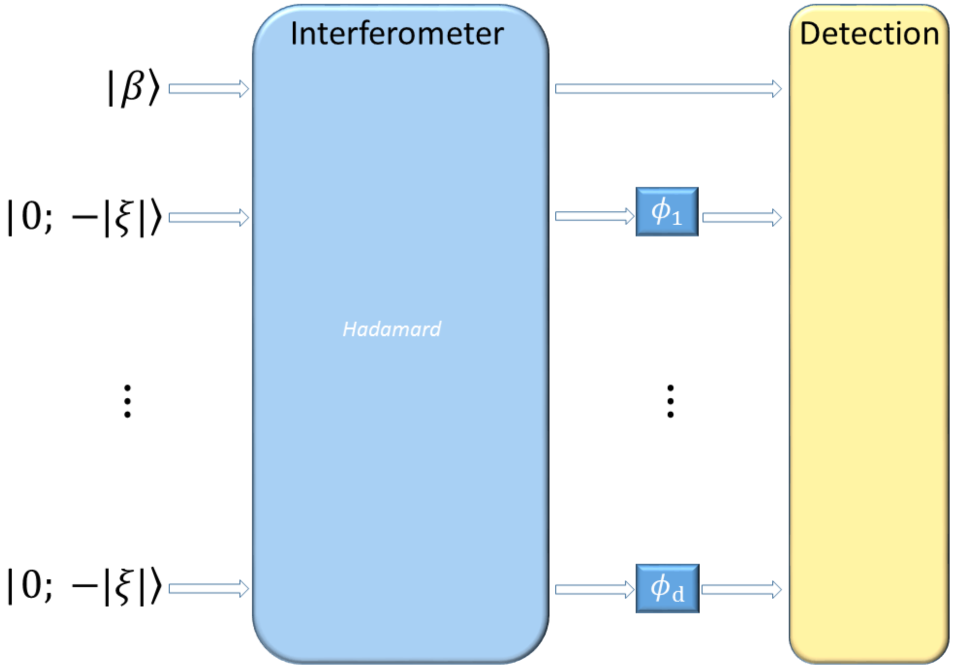
The ratio $R = \frac{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{sim}}^{-1}\right)}{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{ind}}^{-1}\right)} \sim \frac{1}{d}$

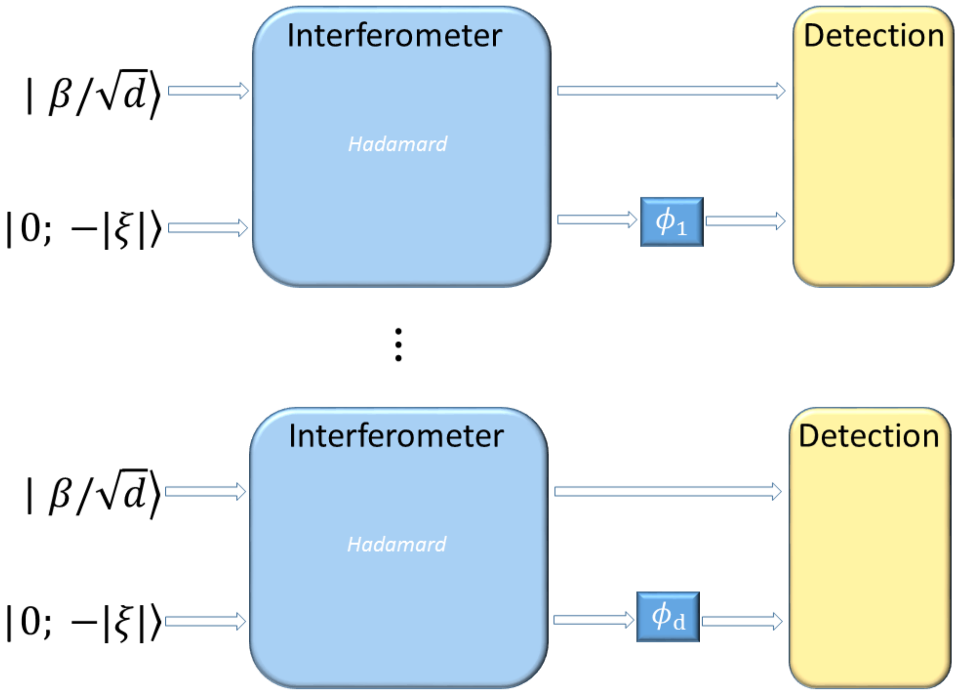
Humphreys et al. Phys. Rev. Lett. **111**, 070403

This reveals a fundamental limitation of Gaussian states when employed in sensing tasks. However, a more general framework is required (left for a *future* work):

- More general estimation problems.
- Is non-Gaussianity a resource? How can we quantify that?

Coherent light and squeezed vacua



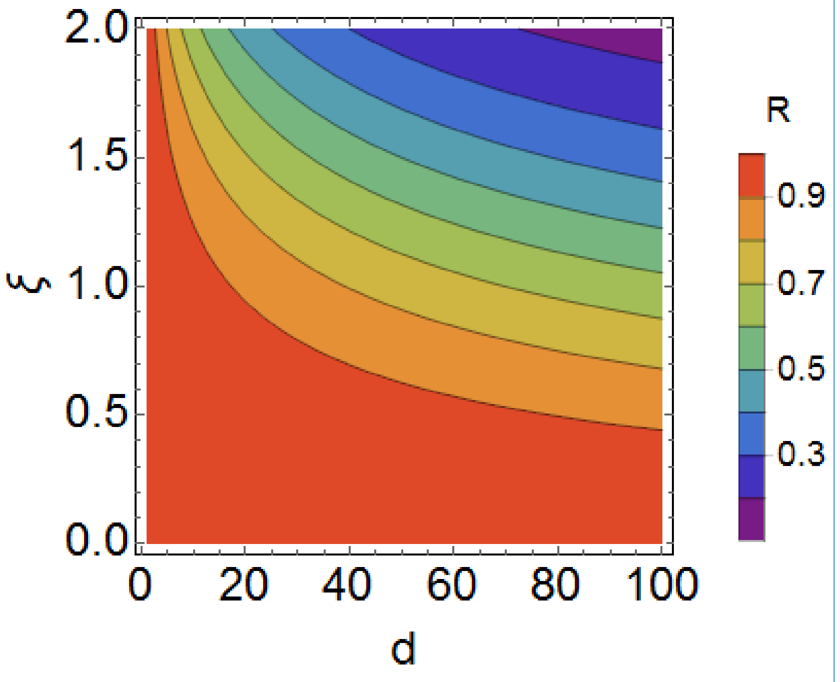


$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \ldots \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

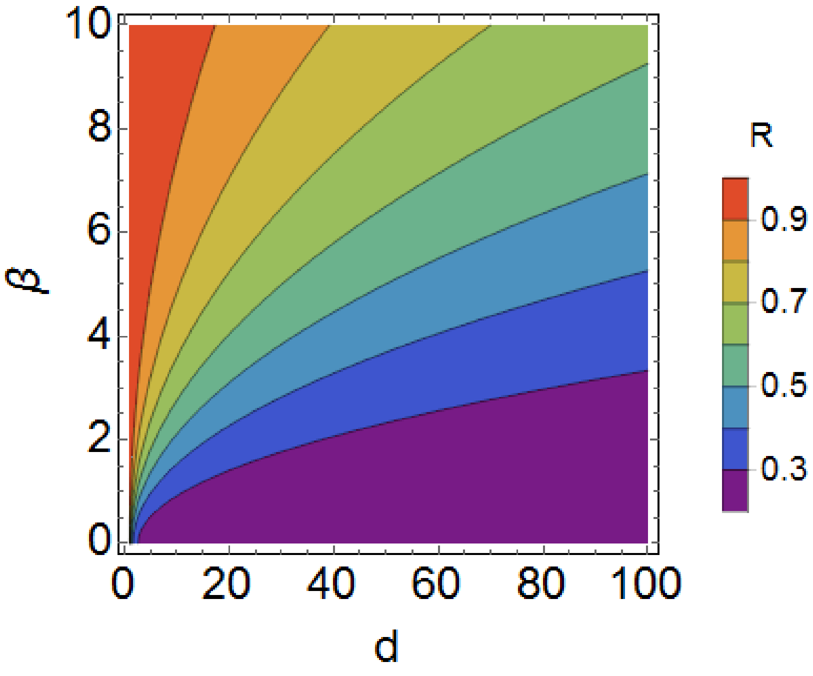
Same input energy in both strategies: the squeezing is the same in both scenarios, the energy conservation comes from the coherent amplitude.

$$\beta \in \Re$$

$$\xi = -|\xi|$$

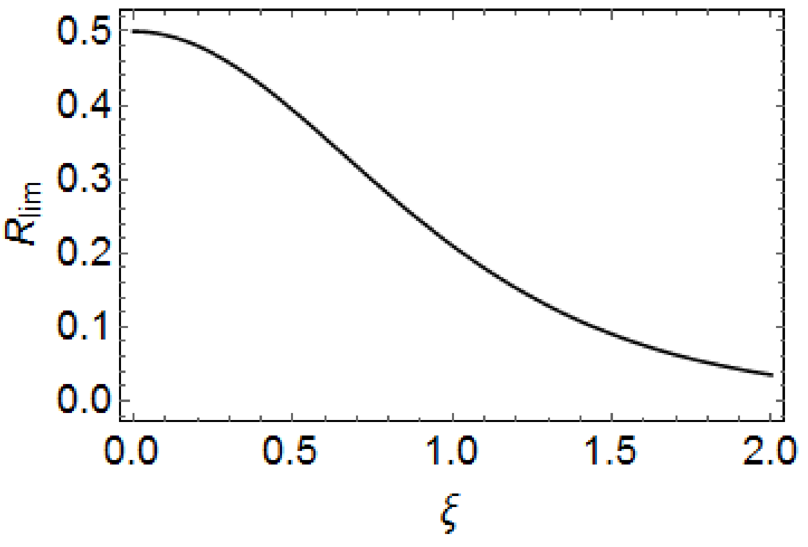


β=10



ξ=1

$$R = 1 - \frac{(d-1)\cosh 2|\xi| \sinh^2 |\xi|}{\beta^2 e^{2|\xi|} + \sinh^2|\xi| + \frac{d-1}{2}\sinh^2 2|\xi|} \leq 1$$



$$d \to \infty$$

$$R_{lim} = \frac{1}{2\cosh^2|\xi|}$$

Goes to zero for large squeezing

Note however that the all-squeezed scenario preforms better

For multiple parameters, a sufficient condition for the saturation is:

$$\mathrm{Tr}\left(\hat{\rho}_{\boldsymbol{\phi}}\left[\hat{\mathcal{L}}_i,\hat{\mathcal{L}}_j\right]\right)=0$$

$\hat{\mathcal{L}}_i$: SLDs

$$\frac{\partial\hat{\rho}_{\boldsymbol{\phi}}}{\partial\phi_i}=\frac{\hat{\mathcal{L}}_i\hat{\rho}_{\boldsymbol{\phi}}+\hat{\rho}_{\boldsymbol{\phi}}\hat{\mathcal{L}}_i}{2}$$

$$\mathbf{H}_{i,j}=\tfrac{1}{2}\mathrm{Tr}\left[\hat{\rho}_{\boldsymbol{\phi}}\left(\hat{\mathcal{L}}_i\hat{\mathcal{L}}_j+\hat{\mathcal{L}}_j\hat{\mathcal{L}}_i\right)\right]$$

Our case: $\frac{\partial\hat{\rho}_{\boldsymbol{\phi}}}{\partial\phi_i}=i\left[(\hat{n}_i-\hat{n}_0),\hat{\rho}_{\boldsymbol{\phi}}\right]\longrightarrow\hat{\mathcal{L}}_i=2i\left[(\hat{n}_i-\hat{n}_0),\hat{\rho}_{\boldsymbol{\phi}}\right]\longrightarrow\mathrm{Tr}\left(\hat{\rho}_{\boldsymbol{\phi}}\left[\hat{\mathcal{L}}_i,\hat{\mathcal{L}}_j\right]\right)=0$

Conclusion - perspectives

Simultaneous better than individual: one reference mode -> more energy (more squeezing = larger variance) in the sensing modes

In the multiple phase estimation modal entanglement is not always helpful, in the sense of comparison simultaneous/individual estimation.

Gaussian states have limitations compared to their non-Gaussian counterparts (e.g. GNS states).

In spite of their limitations, Gaussian states remain practical.

Thank you

Phys. Rev. A **94**, 042342